POSITIVE INTEGERS EXPRESSIBLE AS A SUM OF THREE SQUARES IN ESSENTIALLY ONLY ONE WAY

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ABSTRACT

We consider all positive integers that are expressible as a sum of three squares. For n positive integer, let $P_3(N)$ denote the number of partitions of n as sum of three squares, that is the number (a, b, c) of integers satisfying $n= a^2 + b^2 + c^2$

where $a \geq b \geq c \geq 0$. From these positive integers, we further reduce the numbers and emphasize those that are uniquely represented as a sum of three squares in only one way.

Through the use of imaginary quadratic field properties and Dirichlet class number formula, we are able to deduce that all these n positive integers that are represented in only one way will either be included in the 12 positive integers in the imaginary quadratic field with odd discriminant and small class number of 1 or 2 such that these n’s are element of Z+ and n $\equiv_8$ 3 or be one of the 21 positive integers n such that n is in Z+ and n $\equiv_8$ 1,2,5,6.

With these results, the researcher encourages the readers to pursue similar studies and to look deeper into the two consequences of the Three-Square Theorem like the Theorem of Gauss: Every positive integer n can be expressed as the sum of three triangular numbers and the theorem of Lagrange’s Four-Square Theorem: Every positive integer n can be expressed as a sum of four squares.

Likewise, modeling problems on natural occurrences should be conducted to make mathematics more appealing and practical especially to non-mathematics practitioners.

Keywords: **Positive Integers, Expressible, Essentially, Sum of Three Squares.**

Section 1. Introduction

The set S consisting of those positive integers n which are uniquely expressible in the form $n = a^2 + b^2 + c^2$, $a \geq b \geq c \geq 0$, is considered. Since n ϵ S if and only if 4n ϵ S, we may restrict attention to those n not divisible by 4. Classical formulas and the theorem that there are only finitely many imaginary quadratic fields with given class number imply that there are only finitely many n ϵ S with $n \not\equiv 0 \pmod 4$. More specifically, from the existing knowledge of all the imaginary quadratic fields with odd discriminant and class number 1 or 2 it is readily deduced that there are precisely twelve positive integers n such that n ϵ S and $n \equiv 3 \pmod 8$. To determine those n ϵ S such that $n \equiv 1, 2, 5, 6 \pmod 8$ requires the determination of the imaginary quadratic fields with even discriminant and class number 1, 2, or 4. While the latter information is known empirically, it has not been proved that the known list of 33 such fields is complete. If it is complete, then our arguments show that there are exactly 21 positive integers n such that n ϵ S and $n \equiv 1, 2, 5, 6 \pmod 8$.

1.1 Statement of the Problem

This study aimed to discuss and give a detailed proof of theorem 1 and theorem 2 to show Positive Integers Expressible as a Sum of Three Squares in Essentially Only One Way. Specifically, it aimed to:

1. deduce that there are precisely twelve positive integers n such that n ϵ S and $n \equiv 3 \pmod 8$ from existing knowledge on imaginary quadratic fields with odd discriminant and class number 1 or 2 it is readily; and
2. determine those n ϵ S such that $n \equiv 1, 2, 5, 6 \pmod 8$ requires the determination of the imaginary quadratic fields with even discriminant and class number 1, 2, or 4.

1.2 Significance of the Study

In this paper we discuss one of the consequences of the effective determination of the imaginary quadratic fields with small class number.

If n is a positive integer, let $P_3(n)$ denote the number of partitions of n as a sum of three squares, i.e., the number of triples (a, b, c) of integers satisfying

$$n = a^2 + b^2 + c^2, \quad a \geq b \geq c \geq 0 \qquad (1)$$

Now if a multiple of 4 is expressible as a sum of three squares, all the squares must be even such that $P_3(4n) = P_3(n)$. Also, $P_3(n)$ in our discussion will be assumed that $n \neq 0 \pmod 4$ in effect that $n \equiv 1, 2, 3, 5, 6, 7 \pmod 8$.

1.3 Research Methodology

1.3.1 Research Design

We consider all positive integers that are expressible as a sum of three squares. For a n positive integer, let $P_3(N)$ denote the number of partitions of n as sum of three squares, that is the number (a,b,c) of integers satisfying

$$n = a^2 + b^2 + c^2$$

where $a \geq b \geq c \geq 0$. From these positive integers, we further reduce the numbers and emphasize those that are uniquely represented as sum of three squares in only one way.

Let us show the integers 1-50 as sum of three squares as a basis to illustrate further the content of the paper.

1 = $1^2 + 0^2 + 0^2$	26 = $4^2 + 3^2 + 1^2 = 5^2 + 1^2 + 0^2$
2 = $1^2 + 1^2 + 0^2$	27 = $3^2 + 3^2 + 3^2 = 5^2 + 1^2 + 1^2$
3 = $1^2 + 1^2 + 1^2$	28 = N/A
4 = $2^2 + 0^2 + 0^2$	29 = $5^2 + 2^2 + 0^2 = 4^2 + 3^2 + 2^2$
5 = $2^2 + 1^2 + 0^2$	30 = $5^2 + 2^2 + 1^2$
6 = $2^2 + 1^2 + 1^2$	31 = N/A
7 = N/A	32 = $4^2 + 4^2 + 0^2$
8 = $2^2 + 2^2 + o^2$	33 = $4^2 + 4^2 + 1^2 = 5^2 + 2^2 + 2^2$
9 = $2^2 + 2^2 + 1^2 = 3^2 + 0^2 + 0^2$	34 = $4^2 + 3^2 + 3^2 = 5^2 + 3^2 + 0^2$
10 = $3^2 + 1^2 + 0^2$	35 = $5^2 + 3^2 + 1^2$
11 = $3^2 + 1^2 + 1^2$	36 = $4^2 + 4^2 + 2^2 = 6^2 + 0^2 + 0^2$
12 = $2^2 + 2^2 + 2^2$	37 = $6^2 + 1^2 + 0^2$
13 = $3^2 + 2^2 + 0^2$	38 = $5^2 + 3^2 + 2^2 = 6^2 + 1^2 + 1^2$
14 = $3^2 + 2^2 + 1^2$	39 = N/A
15 = N/A	40 = $6^2 + 2^2 + 0^2$
16 = $4^2 + 0^2 + 0^2$	41 = $4^2 + 4^2 + 3^2 = 5^2 + 4^2 + 0^2$
17 = $3^2 + 2^2 + 2^2 = 4^2 + 1^2 + 0^2$	$= 6^2 + 2^2 + 1^2$
18 = $3^2 + 3^2 + 0^2 = 4^2 + 1^2 + 1^2$	42 = $5^2 + 4^2 + 1^2$
19 = $3^2 + 3^2 + 1^2$	43 = $5^2 + 3^2 + 3^2$
20 = $4^2 + 2^2 + 0^2$	44 = $6^2 + 2^2 + 2^2$
21 = $4^2 + 2^2 + 1^2$	45 = $5^2 + 4^2 + 2^2 = 6^2 + 2^2 + 1^2$
22 = $3^2 + 3^2 + 2^2$	46 = $6^2 + 3^2 + 2^2$
23 = N/A	47 = N/A
24 = $4^2 + 2^2 + 2^2$	48 = $4^2 + 4^2 + 4^2$
25 = $4^2 + 3^2 + 0^2 = 5^2 + 0^2 + 0^2$	49 = $7^2 + 0^2 + 0^2$
	50 = $5^2 + 4^2 + 3^2 = 5^2 + 5^2 + 0^2$
	$= 7^2 + 1^2 + 0^2$

Notice that the squares of integers can either be 0, 1, 4 (mod 8).

Next, is to partition the integers into equivalence classes modulo 8:

[0], [1], [2], [3], [4], [5], [6], [7]

[0] = 8k, k ϵ Z

$8k^2 = 64, k^2 \equiv 0(\text{mod } 8)$

[1] = 8k + 1, k ϵ Z

$(8k + 1)^2 = 64, k^2 + 16k +1 \equiv 1(\text{mod } 8)$

[2] = 8k + 2, k ϵ Z

$(8k + 2)^2 = 64, k^2 + 16k +4 \equiv 4(\text{mod } 8)$

[3] = 8k + 3, k ϵ Z

$(8k + 3)^2 = 64, k^2 + 16k + 9 \equiv 1(\text{mod } 8)$

[4] = 8k + 4, k ϵ Z

$(8k + 4)^2 = 64, k^2 + 16k + 16 \equiv 0(\text{mod } 8)$

[5] = 8k + 5, k ϵ Z

$(8k + 5)^2 = 64, k^2 + 16k + 25 \equiv 1(\text{mod } 8)$

[6] = 8k + 6, k ϵ Z

$(8k + 6)^2 = 64, k^2 + 16k + 36 \equiv 4(\text{mod } 8)$

[7] = 8k + 7, k ϵ Z

$(8k + 7)^2 = 64, k^2 + 16k + 49 \equiv 1(\text{mod } 8)$

Then, we can write the equivalence classes as sum of three squares.

$0 \equiv (\text{mod } 8)$ as 4 + 4 + 0 or 0 + 0 + 0
$1 \equiv (\text{mod } 8)$ as 1 + 0 + 0 or 4 + 4 + 1
$2 \equiv (\text{mod } 8)$ as 1 + 1 + 0
$3 \equiv (\text{mod } 8)$ as 1 + 1 + 1
$4 \equiv (\text{mod } 8)$ as 4 + 0 + 0 or 4 + 4 + 4
$5 \equiv (\text{mod } 8)$ as 4 + 1 + 0
$6 \equiv (\text{mod } 8)$ as 4 + 1 + 1
$7 \equiv (\text{mod } 8)$ as N/A

The integer 7 cannot be written as a sum of three integers from 0, 1, 4, hence, 7 cannot be written as a sum of three squares.

Let us try to show that *n* is expressible if and only if 4n is also expressible.

($\longrightarrow$) If n is expressible then 4n is also expressible.

$$n = a^2 + b^2 + c^2$$
$$4n = 4(a^2 + b^2 + c^2)$$
$$= 4a^2 + 4b^2 + 4c^2$$
$$= (2a)^2 + (2b)^2 + (2c)^2$$

($\longleftarrow$) If 4n is expressible then n is also expressible.

$$4n = x^2 + y^2 + z^2.$$

Since 4*n* will either belong to 0(mod8) or 4(mod8). Observe that the sets of three squares of 0(mod8) and 4(mod8) are all divisible by 4. Thus, we can write it as:

$$n = \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4}$$

$$n = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2$$

Since $x^2 + y^2 + z^2 \in \{0, 4\}$ mod 8, then x, y, z is either in [0], [2], [4], [6] and this implies that x, y, z is even. Therefore x/2, y/2, z/2 QED.

Imaginary Quadratic (IQ) Field

Set of rational numbers where the square of each of the element is irrational.

*Say F is an imaginary quadratic field inside C such that

$F = Q(\sqrt{n}) = a + b\sqrt{n} : a, b \in Q, n \in Z$ except { 0, 1} square free.

Discriminant

If *n* is an element of IQ, then determinant of *n* is *n* if $n \equiv 4 = 1$, Otherwise it is 4*n*.

Consider the imaginary quadratic field F:

$$D\,F = n \quad \text{if } n = 1 \pmod 4$$
$$= 4n \quad \text{if } n = 2 \pmod 4$$

Class Number

$$w(F) = \begin{cases} 4 & \text{if } F = Q(\sqrt{-1}) \\ 4n & \text{if } F = Q(p - 3) \\ 2 & \text{otherwise} \end{cases}$$

Through the use of imaginary quadratic field properties and Dirichlet class number formula, we are able to deduce that all these n positive integers that are represented in only one way will either be included in the 12 positive integers in the imaginary quadratic field with odd discriminant and small class number of 1 or 2 such that these n's are element of Z+ and $n \equiv_8 3$ or be one of the 21 positive integers n such that n is in Z+ and $n \equiv_8 1,2,5,6$.

We denote $r_3(n)$ to be the number of triples (i, j, k) of integers such that:

$$i^2 + j^2 + k^2 = n \qquad (2)$$

with no restriction on the signs or relative sizes of *i*, *j*, *k*. Each solution of (1) gives rise to several solutions of (2) by permutations of the summands and changes of signs.

$\rho(a, b, c) = 6$ if $a > b = c = 0$

$\rho(a, b, c) = 8$ if $a = b = c = 0$

$\rho(a, b, c) = 12$ if $a = b > c = 0$

$\rho(a, b, c) = 24$ if $a > b > c = 0$ or
if $a > b = c > 0$ or
if $a = b > c > 0$

$\rho(a, b, c) = 48$ if $a > b > c > 0$

For $\rho(a, b, c) = 6$ if $a > b = c = 0$

$\rightarrow (\pm a^2 \pm 0^2 \pm 0^2), (\pm 0^2 \pm b^2 \pm 0^2), (\pm 0^2 \pm 0^2 \pm c^2)$

For ρ (a, b, c) = 8 if a = b = c =0

$\rightarrow \pm a^2 \pm a^2 \pm a^2$

For ρ (a, b, c) = 12 if a = b>c=0

$\rightarrow (\pm a^2 \pm a^2 \pm 0^2), (\pm a^2 \pm 0^2 \pm a^2), (\pm 0^2 \pm a^2 \pm a^2)$

For ρ (a, b, c)=24 if a>b>c=0 or a>b= c>0 or a = b>c>0

$\rightarrow (\pm a^2 \pm b^2 \pm 0^2), (\pm a^2 \pm 0^2 \pm b^2), (\pm b^2 \pm a^2 \pm 0^2)$

$\rightarrow (\pm b^2 \pm 0^2 \pm a^2), (\pm 0^2 \pm a^2 \pm b^2), (\pm 0^2 \pm b^2 \pm a^2)$

For ρ (a, b, c) = 48 if a > b > c > 0

$\rightarrow (\pm a^2 \pm a^2 \pm b^2), (\pm a^2 \pm b^2 \pm a^2), (\pm b^2 \pm a^2 \pm b^2), (\pm a^2 \pm b^2 \pm b^2)$

For most values of n, only the last case ρ (a, b, c) = 48 occurs and thus for these values of n the ratio $r_3(n)/P_3(n)$ is exactly 48. In any case, we have

$$r_3(n) \geq 48P_3(n)$$

Now, we introduce another notation for the paper. Let $R_3(n)$ be defined as the number of triples (i, j, k) of integers that is:

$$n = i^2 + j^2 + k^2$$

$$\gcd(i, j, k)$$

We use a formula to get the $r_3(n)$ by using the $R_3(n)$.

$$r_3(n) = \sum_{d^2 | n} R_3(n/d^2),$$

1.4 Definition of Terms

Algebraic number theory. Algebraic number theory is a branch of number theory that uses the techniques of abstract algebra to study integers, rational numbers, and their generalizations. Number-theoretic questions are expressed in terms of properties of algebraic objects such as algebraic number fields and their rings of integers, finite fields, and function fields.

Corollary. Corollary is the name given to a theorem that follows because of another theorem.

Definitions. Definition in mathematics has been the stipulative conception, according to which a definition merely stipulates the meaning of a term in other terms which are supposed to be already well known. The stipulative conception has been so dominant and accepted as unproblematic that the nature of definition has not been much discussed, yet it is inadequate.

Lemmas. Lemmas are baby theorems. It is a name for a theorem that serves as a component or steppingstone to reach the desired (main) result, the main result or result of bringing together of these lemmas usually are named theorems by authors typically.

Number theory. Number Theory (or arithmetic or higher arithmetic in older usage) is a branch of pure mathematics devoted primarily to the study of integers. German mathematician Carl Friedrich Gauss (1777–1855) said, "Mathematics is the queen of the sciences—and number theory is the queen of mathematics."

Square Free Number. A number is said to be square free if no prime factor divides it more than once, i.e., largest power of a prime factor that divides n is one. First few square free numbers are 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39, …

Theorem. Theorems are logically equivalent to the axioms that need to be proven.

Section 2. RESULTS AND DISCUSSIONS

When n is square free, we use the Dirichlet class number formula, i.e.

$$r_3(n) = R_3(n) = 48\frac{h(-n)}{w(-n)} \text{ if } n \equiv 3 \pmod 8$$

$$r_3(n) = R_3(n) = 24\frac{h(-4n)}{w(-4n)} \text{ if } n \equiv 1, 2, 5, 6 \pmod 8$$

where h(d) is class number and w(d) is the number of units of the imaginary quadratic field of discriminant d.

We start by the Legendre's Square Theorem where if $n \equiv_8 3$, then n is expressible as a sum of 3 squares. Also, $n \equiv_8 7$ is not expressible as sum of 3 squares. Moreover, if $n \equiv_8 4$, then n is expressible as sum of 3 squares which implies that $n = 4n_1$, n_1 is expressible as sum of 3 squares and hence, we now have $n \equiv_8 1,2,5,6$ is possible by sum of 3 squares.

Furthermore, we remove integers that can be expressed as sum of 3 squares in more than one way. By the quadratic field, we first get the quadratic fields with even and odd discriminant so that the integers expressible as sum of three squares with perfect squares will be removed."... if d is the of Q(d), then d is non-square..."

LEMMA 1. If $n \equiv_8 1, 2, 3, 5, 6$, then $R_3(n) > 0$,i.e. *n* can be expressed as a sum of three squares without common factor.

Let *x,y,z* be integers. Suppose an integer $n > 4$ is a square-free integer that can be expressed as sum of three square numbers.

The number of solutions of $x^2 + y^2 + z^2 = n$ without restriction on the signs or relative sizes of *x, y,* and *z* is given by $r_3(n)$.

Proof:

$$r_3(n) \begin{cases} = 24h(-n) & \text{for } n \equiv 3 (\text{mod} 8) \\ = 12h(-4n) & \text{for } n \equiv 1,2,5,6 \pmod 8 \\ = 0 & \text{for } n \equiv 7 \pmod 8 \end{cases}$$

Furthermore, we can conclude that if *n* is square-free, $r_3(n) = R_3(n)$ since if $\gcd(x, y, z)=1$, then we cannot write $n = x^2 + y^2 + z^2$ as $n = a^2(x^2 + y^2 + z^2)$ where $(ax)^2$, $(ay)^2$, $(az)^2 = x^2, y^2, z^2$ respectively.

LEMMA 2. If n=1,2,3,5, 6 (mod 8) and n is divisible by p^2, where p is an odd prime, then $P_3(n) > 1$.

Proof:

Through the use of *Lemma 1, we* know that $R_3(n) > 0$, hence we know that there exist integers *x, y, z* where $x^2 + y^2 + z^2 = n$, $x \geq y \geq z \geq 0$, $gcd(x, y, z) = 1$

Now, by assumption that n is divisible by p^2 in effect sing Lemma 1 again,
We have $R_3(^n/_p{}^2) > 0$ and this implies that there exists a, b, c with

$$a^2 + b^2 + c^2 = n,\ a \geq b \geq c \geq 0,\ \gcd(a, b, c) = 1$$

Thus, we can conclude that (x, y, z) and (pa, pb, pc) are different solutions.

LEMMA 3. If n is a square free, then.

$$r_3(n) = R_3(n) = 24h(-n) \quad \text{if } n>3,\ n \equiv_8 3$$
$$r_3(n) = R_3(n) = 12(-4n) \quad \text{if } n>1,\ n \equiv_8 1, 2, 5, 6$$

Proof:

It can be observed that since we are talking about imaginary quadratic field, we have w (d)=2. Now, using the equations

$$r_3(n) = R_3(n) = 48\frac{h(-n)}{w(-n)} \quad \text{if } n \equiv 3 \pmod 8$$

$$r_3(n) = R_3(n) = 24\frac{h(-4n)}{w(-4n)} \quad \text{if } n \equiv 1, 2, 5, 6 \pmod 8$$

We will substitute the w(d)=2, hence get the case for r_3 (n), in effect,

$$r_3(n) = R_3(n) = 24\,h(-n) \quad \text{if } n > 3,\ n \equiv_8 3$$
$$r_3(n) = R_3(n) = 12(-4n) \quad \text{if } n > 1,\ n \equiv_8 1, 2, 5, 6$$

LEMMA 4. If n is a square free and $n \equiv_8 3$, then $h(-n) \leq 2P_3(n)$. If n isa square free and $n \equiv_8 1, 2, 5, 6$, then $h(-4n) \leq 4P_3(n)$

Proof:

We have by Lemma 3 and equation $r_3(n) \geq 48P_3(n)$. Then, we will have

$$24\,h(-n) = r_3(n) \geq 48P_3(n)$$

Now, we can write this as

$$24\,h(-n) \geq 48P_3(n)$$

Simplifying, we will get

$$h(-n) \geq 2P_3(n) \quad \text{for } n \equiv 3 \pmod 8$$

Class Numbers

The case for $n \equiv 1, 2, 5, 6 \pmod 8$ is similar. We have:

$$12\,h(-n) = r_3(n) \geq 48P_3(n)$$
$$12\,h(-n) \geq 48P_3(n)$$
$$h(-n) \geq 4P_3(n)$$

Section 3. SUMMARY AND RESEARCH DIRECTION

Summary

Here is a list of some class numbers:

Class Number 1: 3, 4, 7, 8, 11, 19, 43, 67, 163

Class Number 2: 5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427
Class Number 3: 23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907
Class Number 4: 14, 17, 21, 30, 33, 34, 39, 42, 46, 55, 57, 70, 73, 78, 82, 85, 93, 97, 102, 130, 133, 142, 155, 177, 190, 193, 195, 203, 219, 253

Conclusions

The main result of the paper is the following theorems.

Theorem 1. If $n \equiv_8 3$, then $P_3(n)=1$ if and only if n is one of the twelve numbers as listed: 3, 11, 19, 35, 43, 67, 91, 115, 163, 235, 403, 427

Proof:

If $n \equiv 3 \pmod 8$ but n is not square free, then $P_3(n) > 1$ by lemma 2. If $n \equiv 3 \pmod 8$ and n is square free, then we use Lemma 4 to those with class numbers of 1 or 2.

By previous studies, there are six known square free positive integers n where the class number h(−n) is 1 and $n \equiv 3 \pmod 8$. These are 3,11,19,43,67,163. If n is one of the five numbers $11 = 3^2 + 21^2$, $19 = 2 * 3^2 + 12$, $43 = 5^2 + 2 * 3^2$, $67 = 7^2 + 2 * 3^2$, $163 = 2 * 9^2 + 1^2$, then the first equation has only one solution and $r_3(n) = 24$. Hence, these 5 integers are included in the said 13 integers of the theorem.

Based on previous paper on complex quadratic fields with class number two, there are ten square free positive integers n with $n \equiv 3 \pmod 8$ in effect 35, 51, 91, 115, 123, 187, 235, 267, 403, 427. We can observe that there is exactly one solution for these numbers.

Furthermore, $a > b > c > 0$. This is the last case hence $r_3(n) = 48$. Thus, these 6 integers are also included on the list.

The remaining 4 integers $51=7^2 +2 * 1^2$, $123=11^2 +2 * 1^2$, $187=13^2+2 * 3^2$, $267=13^2 +2 * 7^2$ has $P_3(n)=2$ since $r_3(n)=24$ $P_3(n)=48$ meaning they can be written as sum of three squares in more than one way. Hence, they are discarded.

Therefore, we only have twelve integers: 3, 11, 19, 35, 43, 67, 91, 115, 163, 235, 403, 427 that can be essentially written in one way if $n \equiv 3 \pmod 8$.

Therefore, Theorem1 has been proved.

Theorem 2. If $n \equiv_8 1, 2, 5, 6$ and if $P_3(n) = 1$, then neither (a) n is one of the twenty-one number listed below or (b) n>106, n is square free, and h(−4n) = 4. (1, 2, 5, 6, 10, 13, 14, 21, 22, 30, 37, 42, 46, 58, 70, 78, 93, 133, 142, 190, 253)

Proof:

If $n \equiv 1,2,5,6 \pmod 8$ but n is not square free, then $P_3(n)$ by lemma 2. If $n \equiv 1,2,5,6 \pmod 8$ and n is square free, using lemma 4 implies that we only need to look at n with $h(-4n) \leq 4$.

By previous studies, the only square free positive integers n with $n \equiv 1,2,5,6 \pmod 8$ and h(−4n) = 1 are n = 1,2. Obviously $P_3(1) = p_3(2) = 1$.

Also, there are exactly seven square free positive integers n which $n \equiv 1,2,5,6 \pmod 8$ and h(−4n) = 2, namely, $5 = 2^2 + 1^2$, $6 = 2^2 + 2 * 1^2$, $10 = 3^2 + 1^2$, $13=3^2 + 2^2$, $22 =2. 3^2 +2^2$, $37 = 6^2 + 1^2$, $58 = 7^2 + 3^2$.

There are no square free positive integers n such that $n \equiv 1,2,5,6 \pmod 8$ and h(−4n) = 3.

Also, there are exactly 24 square free positive integers such that $n \equiv 1,2,5,6 \pmod 8$, $h(-4n) = 4$ and $1 < n < 10^6$. Twelve (12) of these values of n are $14=3^2+2^2+1^2$, $21 = 4^2+2^2+1^2$, $30 = 5^2+2+1^2$, $42 = 5^2+4^2+1^2$, $46 = 6^2+3$, 70, 78, 93,133, 142, 190, 253.

For these values of n, we have $r_3 = 48$ then by Lemma 4, $P_3(n) = 1$.

The other 12 are $17 = 3^2+2^2+2^2$, $33 = 5^2+2^2+2^2$, $34 = 5^2+3^2+0^2$, 57, 73, 82, 85, 97, 102, 130, 177, 193.

For these integers, $r_3(n) = 24$, then by Lemma 4 we know that $r_3(n) = 24P_3(n) = 48$, then $P_3(n) = 2$. Thus, Theorem 2 was proven.

Recommendations

With these results, the researcher encourages the readers to pursue similar studies and to look deeper into the two consequences of the Three-Square Theorem like the Theorem of Gauss: Every positive integer n can be expressed as the sum of three triangular numbers and the theorem of Lagrange's Four-Square Theorem: Every positive integer n can be expressed as a sum of four squares.

Likewise, modeling problems on natural occurrences should be conducted to make mathematics more appealing and practical especially to non-mathematics practitioners.

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