Evaluation of Formal-Rhetorical and Problem-Centered Mathematical Proof of Students

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Abstract

This is the result of a study aimed at evaluating the process of verification of student mathematics education in performing of proof using of the formal-rhetorical part and problem-centered part as proof structure. Description of a combination of the understanding of the formal-rhetorical part and problem-centered part in proving the lemma, theorem and the corollary in Real Analysis will bring the creative side of the students in understanding and validating as well as constructing proof. The formal-rhetorical part sometimes said to be a proof of proof framework, while the problem-centered part relying purely on mathematical problem solving, intuition, and understanding that are more related to the concept. Selden and Selden (2013) stated that two aspects of the structure of this evidence is proof genre.

Keywords: Proof, the formal-rhetorical part, the problem-centered part.

A. Introduction

Since the end of the nineteen eighties, a mathematical proof and evidence is one of the main issues of research in mathematics education, and this is a different mapping of the research that has been developed at that time. In mathematics, proof is a series of logical arguments that explain the truth of a statement, which is logical here are the steps at each argument must be justified by the previous step. According to Healy and Hoyles (Cheng & Lin, 2009, p. 124) proof is at the heart of mathematical thinking and deductive reasoning, while according to Chen (2008, p. 398) a proof is a step-by-step demonstration that a statement is valid. Selden dan Selden (Lee & Smith, 2009, p. 21) proof can be considered asa special form of argumentation in which deductive logic acts as a norm of warranting mathematical assertions. Furthermore Mariotti (2006, p. 189) defines proof as a series of logical implication that generates theoretical validation of a statement.

In mathematics education has been considered that there are at least three kinds of aspects of the proof when proving something that is: (1) be aware of when proof can be seen as an object of structural, which consists of the following components: a proposition certain, a proposition universal and deductive reasoning, (2) realized when proof can be seen as an intelligence activity. Evidence as activities allow clarification of what proof supports the activities and involved in these activities, (3) realized when evidence can be seen as the role and function in mathematics, empirical science and the real world. Therefore, since the proof can be a driving force in productive activities throughout the lecture a student, then they can appreciate the true meaning and importance of evidence through the lecture (Yumoto and Miyasaki, 2009, p. 76-77).

There is no denying that the process of proof in mathematics is a complex matter involving various competencies pupil / student, including identifying assumptions, sorting out the of properties and structures, as well as arranging / composing each argument to be logical and valid. So the ability of students to prove valid statement is the key to success in mathematics.

To assist the difficulties often experienced by students in writing proof, Selden and Selden (2013, p. 308) states that students needs to be assisted by applying two aspect/part of the evidence that is: (1) *The formal-rhetorical part*, this part is sometimes called the proof framework. Part of the evidence depends only on the base and use the logical structure of the statement of theorems, definitions related, and previous results. In general, this section does not depend on a deep understanding, or intuition about the concepts involved or problem to be solved (Schoenfeld, 1985, p. 74); (2) *The problem-centered part*. Part of this depends purely on mathematical problem solving, intuition, and understanding more concerned with concepts (Selden & Selden, 2009).

To understand these two aspects raised by Selden and Selden, an example is given as follows:

Example: Suppose that $\{x_n\}$ is a convergent sequence and $\{y_n\}$ is such that for any $\varepsilon > 0$, there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that $\{y_n\}$ is convergent? (Bartle & Sherbert. Exercise 3.2, No.22, 2010).

Proof:

Take any $\varepsilon > 0$, and because $\{x_n\}$ is convergent, in other words $\{x_n\} \rightarrow x$, $\exists\, M_1 \in \mathrm{N}$, such that $|x_n - x| <$ ______ $, \forall\, n \geq M_1$.

Based on the assumption, $\exists\, M_2 \in \mathrm{N}$, such that $|x_n - y_n| <$ ______ , $\forall\, n \geq M_2$. By Using the triangle inequality properties,

$|y_n - x| =$ |__________________|

= |________ + ________| ≤ |______| + |______| < __ + __ < ε,

$\forall\, n \geq M = \max\,\{M_1, M_2\}$.

Therefore $\{y_n\} \rightarrow x$, it means that the sequence $\{y_n\}$ is convergent.

The formal-rhetorical part often called the framework of proof outlined above, might be helpful for students to write their appropriate proof norms in the mathematical community, and it is in need of technical skills, but resolve the problem on the pieces proof that this is a part that emptied into the central problem. Not as easy as imagined by students in filling the void from the proof on Real Analysis, may be students who have experience and a good understanding, which will be able to fill the void of a piece of evidence by example $\frac{\varepsilon}{2}$, using mathematical manipulations and utilize the triangle inequality properties, as well as $M = \max \{M_1, M_2\}$.

In mathematics, arguments and proof can be explained by four functional characteristics described of the common aspects between them, ie (Pedemonte 2007, p. 26):

(1) The arguments and proof in mathematics is a rational justification

Characteristics of justification is seen in the form of argument: reasoning, an explicit conclusions derived from one or more statements given (Duval, 1995). This conclusion is based on the rationality of such conclusions used in juridical language (Plantin, 1990). Juridical model of linguistic theory considers as a model for the argument that confirms the importance of rationality in argumentation (Perelman and Olbrechts-Tyteca, 1958; Toulmin, 1993). In this case, the argument can be considered as the renewal of the Aristotelian rhetoric, but actually by Toulmin (1993), the theory of argumentation nearer to the Aristotelian dialectic. So the argument in mathematics as a proof closer to the dialectic, because it must generate the correct statement.

(2) The arguments and proof in mathematics to convince.

From the standpoint of epistemological, arguments and proof in mathematics developed when someone wants to convince (themselves or others) about the truth of the statement (Chazan, 1993; De Villiers 1990; Hanna 1989, Healy dan Hoyles 2000; Lakatos, 1976). In this context, it is important to distinguish between the terms convincing and persuading, because it is very different in meaning. According to linguistic theory, the aim of convincing is to modify the opinions and beliefs with interesting rationality, while persuading goal is to get the approval without pulling rationality. Convincing persuade states to persuade but by no means assured, resulting in mathematics using only convincing argument.

(3) The arguments and proof in mathematics addressed to a universal audience

If the purpose of argument in mathematics is to convince yourself or the audience about the truth of a statement, then the audience should be able to answer. In linguistic theory, this audience is called universal audience (Plantin, 1990). Audience consists of the mathematical community, classroom, teacher, friend speak.

(4) The arguments and proof in mathematics included in *'field'* (Toulmin, 2003, p. 2)

Linguistic theory states that the meaning of an argument can be different according to the situation of discourse. More specifically, the words can not guarantee an accurate understanding

(Ducrot et al, 1980). It is necessary to look at the proposition, in another context information that allows misunderstandings to be reduced. Character diversity of argument is underlined by the Toulmin (2003), which indicates "*field*" as the idea. For proof is a theoretical field: algebra, calculus, geometry, etc. Field argument in mathematics is limited by the validity criteria. For example, the axiom for the truth value of an argument in a different geometry of the axiom that used in argument algebra.

This is according to Mejia-Ramos (2008) (Mejia-Ramos et al, 2011, p. 334) argues that there are three main argumentative activities related to proof: building a new argument, presenting the arguments provided, and read the arguments given.

Furthermore Soemantri (2004, p. 5.2) explains that the form of the argument is the exceptional circumstances a statement form so too the arguments are exceptional circumstances a statement. So the argument form is defined as follows: If $p_1, p_2, \cdots, p_n$ and q a statement of the conditional $(p_1 \wedge p_2, \cdots, \wedge p_n) \Rightarrow q$ is called form of argument. Any statement $p_1, p_2, \cdots, p_n$ is called the premise, and q the conclusion is called the argument form. While the definition of the argument that if it takes place p, q and so on in the form of an argument has been replaced with a certain statement, then the result is called an argument. Thus, the argument is a conditional statement, the example of a form of argument. The argument is often written as follows:

If Sari grade, so she bought a violin.
Sari grade

Therefore, Sari bought violin

The above argument is an example of a form of argument:

$p \Rightarrow q$

p

So, q

Or $\left[\left(p \Rightarrow q\right) \wedge p\right] \Rightarrow p$

So formal arguments studied in formal logic (historically called symbolic logic, more commonly referred to as mathematical logic) and are presented in a formal language.

B. Research methods

The study was based on a qualitative analysis of the results of tests conducted on 43 students who has programmed courses Real Analysis II. The focus of the evaluation is based on the work of the students in conducting evidentiary based proof structure on formal-rhetorical part and problem-centered part than six (6) given problem. In identifying the understanding of the formal-rhetorical part and problem centered part on problem presented using five categories adopted from (Stylianides, 2009, p. 245) as follows: The argument is valid, logically connected between facts with elements of conclusions will be proved (A1) , argument is valid but not proof (A2), not succeeded in getting into a valid argument (that is, the argument is not valid or unfinished) (A3), argument empirical (A4), and argument-native (ie, responses showed minimal involvement , responses that are not relevant or potentially relevant response but the relevance is not made clear by the conduct of proof) (A5).

C. Discussion

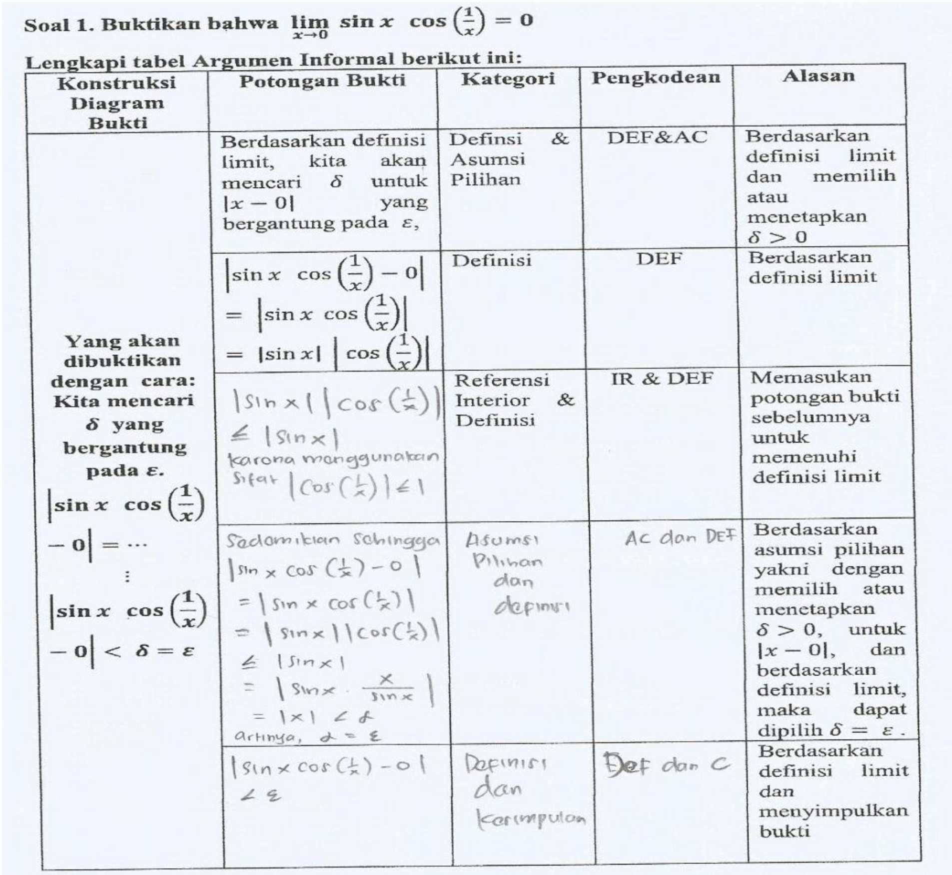
Table 1 shows that the results of a summary of activities to understand and construct the evidence is evaluated according to five categories described earlier show that the arguments that are used both in the section (RF) or (PC) on the work of the students use the argument of non-genuine, or 8 students respond irrelevant in proving.

Table 1. Distribution of lecturers responses to the mathematical proof

Structural proof	Category perspective arguments on proof				
	A_1	A_2	A_3	A_4	A_5
The formal-rhetorical part (RF)	10	2	2	1	3
The problem-centered part (PC)	15	3	1	1	5
Total	25	5	3	2	8

From the evaluation, it appears that 25 students or 60% of the students could make the argument into proof, or in other words, as many as 10 students were able to construct an argument into proof and 15 students were able to validate the argument into proof. But still, there are 18 students or 40% of the students have limited ability to make the argument into a mathematical proof, or 18 students in constructing a proof of formal proof is still experiencing difficulties. Difficulties in understanding the arguments included in the category argument is valid but not proof (A2) as much as 5 students, not succeeded in getting into a valid argument (that is, the argument is not valid or unfinished) (A3) as much as 3 students, argument empirical (A4) as much as 2 students, and argument-native (ie, responses showed minimal involvement , responses that are not relevant or potentially relevant response but the relevance is not made clear by the conduct of proof) (A5) as much as 8 students.

The following will be presented the findings of the work of the students as follows:

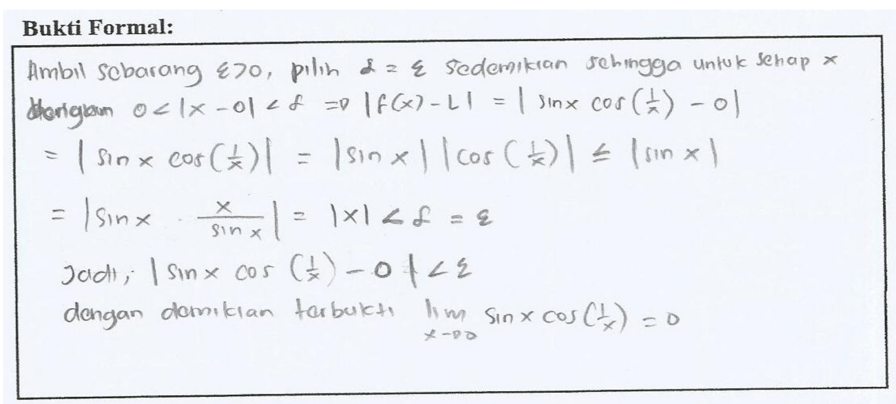


Soal 1. Buktikan bahwa $\lim_{x\to 0} \sin x \cos\left(\frac{1}{x}\right) = 0$

Lengkapi tabel Argumen Informal berikut ini:

Konstruksi Diagram Bukti	Potongan Bukti	Kategori	Pengkodean	Alasan
Yang akan dibuktikan dengan cara: Kita mencari δ yang bergantung pada ε. $\left\|\sin x \cos\left(\frac{1}{x}\right) - 0\right\| = \cdots$ ⋮ $\left\|\sin x \cos\left(\frac{1}{x}\right) - 0\right\| < \delta = \varepsilon$	Berdasarkan definisi limit, kita akan mencari δ untuk $\|x - 0\|$ yang bergantung pada ε,	Definsi & Asumsi Pilihan	DEF&AC	Berdasarkan definisi limit dan memilih atau menetapkan $\delta > 0$
	$\left\|\sin x \cos\left(\frac{1}{x}\right) - 0\right\| = \left\|\sin x \cos\left(\frac{1}{x}\right)\right\| = \|\sin x\| \left\|\cos\left(\frac{1}{x}\right)\right\|$	Definisi	DEF	Berdasarkan definisi limit
	$\|\sin x\| \left\|\cos\left(\frac{1}{x}\right)\right\| \le \|\sin x\|$ karena menggunakan sifat $\left\|\cos\left(\frac{1}{x}\right)\right\| \le 1$	Referensi Interior & Definisi	IR & DEF	Memasukan potongan bukti sebelumnya untuk memenuhi definisi limit
	Sedemikian sehingga $\left\|\sin x \cos\left(\frac{1}{x}\right) - 0\right\| = \left\|\sin x \cos\left(\frac{1}{x}\right)\right\| = \|\sin x\|\left\|\cos\left(\frac{1}{x}\right)\right\| \le \|\sin x\| = \left\|\sin x \cdot \frac{x}{\sin x}\right\| = \|x\| < \delta$ artinya, $\delta = \varepsilon$	Asumsi Pilihan dan definisi	AC dan DEF	Berdasarkan asumsi pilihan yakni dengan memilih atau menetapkan $\delta > 0$, untuk $\|x - 0\|$, dan berdasarkan definisi limit, maka dapat dipilih $\delta = \varepsilon$.
	$\left\|\sin x \cos\left(\frac{1}{x}\right) - 0\right\| < \varepsilon$	Definisi dan kesimpulan	Def dan C	Berdasarkan definisi limit dan menyimpulkan bukti

(a)



Bukti Formal:

Ambil sebarang $\varepsilon > 0$, pilih $\delta = \varepsilon$ sedemikian sehingga untuk setiap x dengan $0 < |x - 0| < \delta \Rightarrow |f(x) - L| = \left|\sin x \cos\left(\frac{1}{x}\right) - 0\right|$

$= \left|\sin x \cos\left(\frac{1}{x}\right)\right| = |\sin x| \left|\cos\left(\frac{1}{x}\right)\right| \le |\sin x|$

$= \left|\sin x \cdot \frac{x}{\sin x}\right| = |x| < \delta = \varepsilon$

Jadi, $\left|\sin x \cos\left(\frac{1}{x}\right) - 0\right| < \varepsilon$

dengan demikian terbukti $\lim_{x\to 0} \sin x \cos\left(\frac{1}{x}\right) = 0$

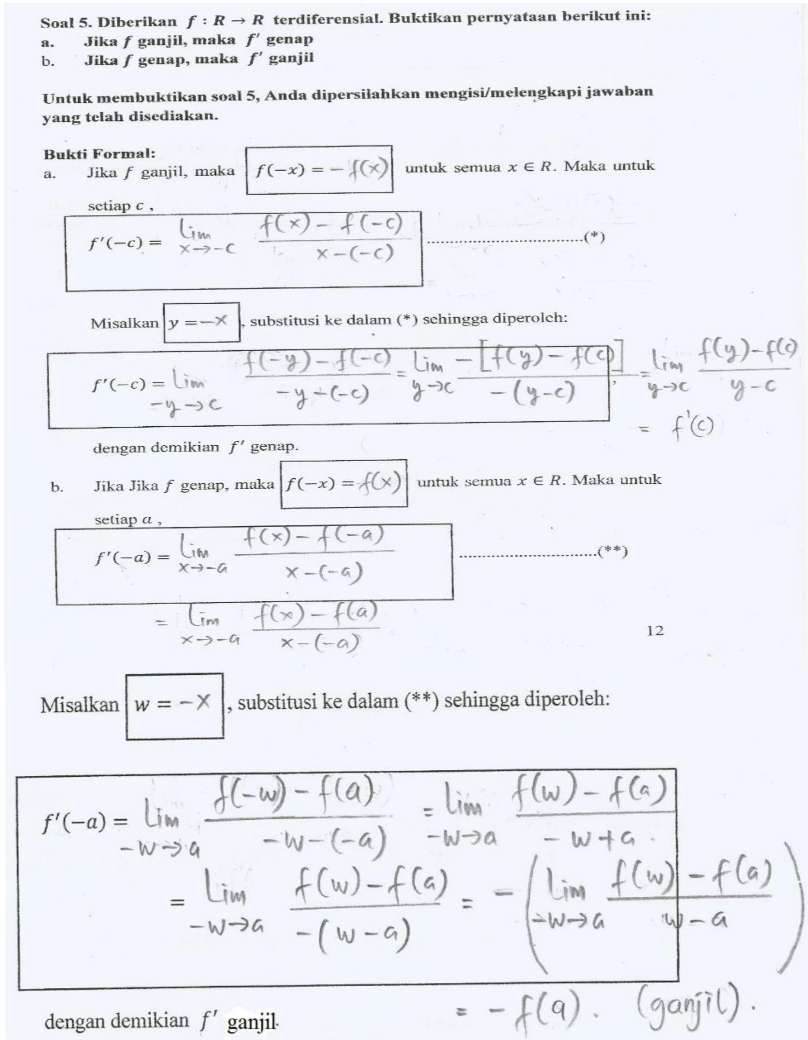
(b)

Figure 1. Sample Results for the Student Employment Problem No. 1

Based on Figure 1 (a) the third row second column, students have been trying to put the pieces proof prior to the categories and coding, but the pieces of evidence in the form of an argument that has not vouch for the validity of the following properties, students have not been using the nature of $|\sin x| \leq |x|$. Further to the next column, the students have taken advantage of the previous step, but the pieces of evidence for the category assumption choice (AC) and (DEF) ie " $\cdots \leq |\sin x| = \left|\sin x \cdot \frac{x}{\sin x}\right| = |x|$" be illogical, although in the choice $\delta = \varepsilon$ as in that column indicates that the student had the right to vote. Furthermore, in preparing the formal proof, step on the first line to the third line students have been doing it right, by utilizing the triangle inequality and properties $\left|\cos\frac{1}{x}\right| \leq 1$, but on the fourth line reoccur students make the mistake of include the "$\cdots \leq |\sin x| = \left|\sin x \cdot \frac{x}{\sin x}\right| = |x|$"as in Figure 1 (b), so that this formal proof to be invalid. This is consistent with the explanation Inglis and Alcock (2012) which states that when reading proof, the students do with reading line by line until the conclusion. They do not try to read the proof structures and techniques used to prove, as a result the idea of proof was not captured properly.

Based on analysis of student work to Question 1, can be found a few mistakes that lead to difficulty in proving the student, as follows:

1. At the time of completing the table informal argument, the students can not yet take advantage of the general nature of $|\cos x| \leq 1$ and $\left|\frac{\sin x}{x}\right| \leq 1$
2. Made a mistake in selecting and manipulating properties inequality $\leq$ into $<$ (mistakes manipulate algebraic form).
3. Students take advantage of the difficulties associated with the concept of a matter to be proved.
4. Students difficulty connecting informal arguments and write back into formal proof.



Soal 5. Diberikan $f : R \to R$ terdiferensial. Buktikan pernyataan berikut ini:
a. **Jika f ganjil, maka f' genap**
b. **Jika f genap, maka f' ganjil**

Untuk membuktikan soal 5, Anda dipersilahkan mengisi/melengkapi jawaban yang telah disediakan.

Bukti Formal:
a. Jika f ganjil, maka $f(-x) = -f(x)$ untuk semua $x \in R$. Maka untuk setiap c,

$f'(-c) = \lim_{x \to -c} \frac{f(x) - f(-c)}{x - (-c)}$ ....(*)

Misalkan $y = -x$, substitusi ke dalam (*) sehingga diperoleh:

$f'(-c) = \lim_{-y \to c} \frac{f(-y) - f(-c)}{-y - (-c)} = \lim_{y \to c} \frac{-[f(y) - f(c)]}{-(y-c)} = \lim_{y \to c} \frac{f(y) - f(c)}{y - c} = f'(c)$

dengan demikian f' genap.

b. Jika Jika f genap, maka $f(-x) = f(x)$ untuk semua $x \in R$. Maka untuk setiap a,

$f'(-a) = \lim_{x \to -a} \frac{f(x) - f(-a)}{x - (-a)}$ ....(**)

$= \lim_{x \to -a} \frac{f(x) - f(a)}{x - (-a)}$

12

Misalkan $w = -x$, substitusi ke dalam (**) sehingga diperoleh:

$f'(-a) = \lim_{-w \to a} \frac{f(-w) - f(a)}{-w - (-a)} = \lim_{-w \to a} \frac{f(w) - f(a)}{-w + a}$

$= \lim_{-w \to a} \frac{f(w) - f(a)}{-(w - a)} = -\left(\lim_{-w \to a} \frac{f(w) - f(a)}{w - a}\right)$

dengan demikian f' ganjil. $= -f(a)$. (ganjil).

Figure 2. Sample Results for the Student Employment Problem No. 5

Based on the results of student work, then it can be outlined ways students can associate the data and facts in the first row, which further complete the definition of $f'(-c)$ as in Figure 2 shows that the students are already familiar with the notion of the odd function and pieces of concept differentiation function at a point $-c$. Furthermore, for the third and fourth lines of the work of students in manipulating algebraic operations so it can be concluded "If f odd, then f' even" shows that there are errors made by the students "$f'(-c) = \lim_{-y \to c} \frac{f(-y) - f(-c)}{-y - (-c)} = \cdots$ ", student analogize the previous step "limit $x \to -c$ with $-y \to c$ ", which should be $-y \to -c$ therefore $y \to c$, however continuation of the work of students on this issue indicate the completion of the process undertaken by students is correct.

To proof completion statement " If f even, then f' odd", ways in which students in the fifth row and the sixth, the students had no trouble started with data and facts about the even function, as well as pieces of concept differentiation function at a point $-a$. But on the eighth row, students returned to make mistakes by inserting "limit $x \to -a$ with $-w \to a$ " and this step continues until the conclusion without altering "limit $-w \to a$ with $w \to a$ " , thus the meaning of this statement be incompatible.

Based on analysis of student work to Question 5, can be found a few errors that lead to difficulty in proving the student, as follows:

1. Students still make a mistake in linking the definition of a derivative with the concept of limit.
2. Students make mistakes utilizing an odd or even function definition.
3. Students make mistakes in substitution for boundaries limit.

D. Conclusions and recommendations

The results provide a significant contribution in evaluating the difficulty of students in formulating arguments logically connected between the facts with the elements of conclusions (A_1), the argument is valid but not proof (A_2), not succeeded in getting into a valid argument (that is, the argument is not valid or unfinished) (A_3), empirical argument (A_4), and the argument is non-genuine (ie, responses showed minimal involvement, response is not relevant, or the response of potentially relevant but the relevance is not made clear by the conduct of proof) (A_5) particularly in understanding and constructing a mathematical proof by *formal-rethorical part* (FC) and *problem centered part* (PC), results of the evaluation showed that although there are 60 % student in category A_1, however they also contained 40% of students are included in the category A_2, A_3, A_4, and A_5 this means that students should always be trained in understanding the evidence and construct proof by using parts FR and PC. The study provides an opportunity for further research with the same or different concepts with a variety of approaches, methods and strategies to understand and construct proof, especially in the subject of Real Analysis.

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