Analysis of the Proof Processes of Pre-Service Teachers regarding Function Concept

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Abstract

The aim of this study is to analyze the proof processes of pre-service teachers regarding the function concept. The study was conducted with six pre-service teachers who were studying at the department of secondary mathematics teaching and who were willing to participate in the study. The data of the study were obtained via semi-structured interviews conducted with the pre-service teachers. It was found that the difficulties experienced by the pre-service teachers in proofs were regarding how to begin the proof process, the use of mathematical language and notations, the use of definitions, forming the setup of the proof and selecting elements from the set.
Key Words: *Proof Processes, Pre-service Teachers, Function Concept*

Introduction

Mathematics emerged as a result of people's mental activities, and it generated its products by way of thinking (Güler & Dikici, 2012). Mathematical proofs, which can be listed among the greatest products of mathematics, make us understand not only that a statement is true but also why it is true. When considered from this perspective, proofs contribute to the systematization and development of mathematics by associating the results with each other (Hanna & Barbeau, 2008). In recent years, subjects such as thinking, proof and reasoning have come into prominence in mathematics instruction research (Heinze & Reis, 2003). This is because one of the most important goals of mathematics instruction is to ensure the development of obtaining logical answers to *'why'* and *'for what'* questions, that is to say, the development of reasoning (Altıparmak & Öziş, 2005). For this reason, a lot of research has been conducted in order to identify the proof-related opinions, beliefs and proof processes of students, pre-service teachers and teachers (Almeida, 2000; 2003; Bayazıt, 2009; Baştürk, 2010; Güler, 2013; Güler & Dikici, 2013; Harel & Sowder, 1998; Jones, 2000; Knuth, 2002; İmamoğlu, 2010; İskenderoğlu, 2010; Raman, 2002; 2003; Recio & Godino, 2001; Stylianides, Stylianides & Philippou, 2007; Weber, 2001).

When the studies in the literature are examined, it is observed that students and pre-service mathematics teachers from all levels experience difficulty in understanding and forming mathematical proof, even though the importance of proof in mathematics and mathematics instruction has been emphasized by many researchers and curriculums (Almeida, 2000; Arslan 2007; Arslan & Yıldız, 2010; Coşkun, 2009; Güler, 2013; Güler & Dikici, 2013; Harel & Sowder, 1998; Jones, 2000; Knapp, 2005; Moore, 1990; 1994; Weber, 2001).

The difficulties experienced by students in performing proof were determined as follows: perceptions regarding the nature of proof (Schoenfeld, 1985), logic and methods of proof (Solow, 1990), problem solving skills (Schoenfeld, 1985) , mathematical language (Laborde, 1990) and concept understanding (Dubinsky & Lewin, 1986; Tall & Vinner, 1981; Vinner & Dreyfus, 1989). Additionally, Moore (1994), grouped the difficulties experienced by students in proof process under seven headings by benefiting from the existing difficulties within the literature. These difficulties are as follows: failure to state the definitions; failure to understand the meanings of the concepts intuitively; failure to use the concept images while performing proof; lack of generalization and use of examples; failure to know what kind of proof structure to use from the definitions; failure to understand the mathematical language and notations; and failure to know how to begin the proof process. Güler & Dikici (2013) classified the difficulties experienced by pre-service teachers regarding algebra in five categories, namely determining how to begin the proof process, stating the definitions, proof representation, the use of mathematical language and notations and the use of logic and methods of proof. However, students are advised to begin proof processes starting from the early years of their educational lives by the National Council of Teachers of Mathematics (NCTM, 2000). Furthermore, "reasoning and proof standard" are among the Five Process Standards within the Principles and Standards for School Mathematics in NCTM (2000) report. Within the scope of this standard, curriculums from the period before pre-school to the 12th grade must ensure the following acquisitions for all students: considering reasoning and proof as the basic components of mathematics, forming and examining mathematical hypotheses, improving and evaluating mathematical claims and proofs, and selecting and using various types of proof methods and reasoning (Dede, 2012).

When the curriculums prepared by the Ministry of National Education [MNE] in our country are examined, it is difficult to speak of an evident proof process in the second level of elementary education (6th-8th grades) due to the lack of concepts like theorem and axiom. It is clearly observed that one of the general aims of the current elementary mathematics curriculum is "to infer about logical inductions and deductions" (MNE, 2005a). Furthermore, "problem solving and mathematical process skills (communication, reasoning and associating)" are listed among the basic skills to be gained within the general objectives of secondary mathematics course (5th-8th grades) curriculum that was renewed by the Ministry of National Education. Accordingly, instead of performing proof in a systematized manner at elementary education level, the emphasis is given to the process of obtaining new information using the distinctive tools (symbols, definitions, relationships, etc.) and thinking techniques (induction, deduction, comparison, generalization, etc.) of mathematics in view of the available information as a result of problem solving skill and thinking (reasoning) (MNE, 2013a).

When the secondary mathematics curriculum is examined, we encounter two acquisitions on the concept of proof in "Logic Learning Area" and "Proof Methods Sub-Learning Area". These are the statements, "they explain the concepts of definition, axiom, theorem and proof, and they state the hypothesis and provision of a theorem" and "they perform simple proofs using the methods of proof" (MNE, 2005b). On the other hand, the skills of "mathematical reasoning and proof" are observed in "Mathematical Process Skills" among the mathematical efficacy and skills that the secondary mathematics course curriculum, which was renewed by the Ministry of National Education, aims to give students. Moreover, we encounter two acquisitions in the renewed curriculum: "acquiring the skills of proof, proportional reasoning and probabilistic thinking" at the 9th grade level and "performing proofs by using mathematical proof methods (giving contrary examples, contrary inverse, direct proof, conflict and induction" at the 11th grade level (MNE, 2013b).

Pre-service teachers' proof skills on the concepts that they will teach in the future need to be researched in detail considering the difficulties experienced by the pre-service teachers regarding mathematical proof and the importance of proof in mathematics instruction. Therefore, the aim of this study is to research pre-service mathematics teachers' proof skills regarding the function concept. By doing so, the source of the difficulties experienced by the pre-service teachers will be revealed more clearly, and accordingly, a benefit will be provided for eliminating the difficulties experienced by the pre-service teachers regarding proof process. Although studies on mathematical proof are common abroad, it can be stated that the studies in this field have become common during the last decade and that there is not an adequate number of studies in this field (Güler & Dikici, 2012). Therefore, this study is believed to contribute to the literature and constitute a reference for prospective studies.

Method

The case study model, which is among the qualitative research methods, was used in the research. Case study is an empirical research method that examines a current phenomenon in its real-life framework and is used when the boundaries between that phenomenon and the content in which it is present are not clear with distinct lines and in cases where there is more than one proof or data source (Yıldırım & Şimşek, 2008).

Pilot Study

Information on the conducted pilot study will be given in this section of the research. The aim of the pilot study is to determine the theorems to be asked in the interviews and the potential difficulties experienced by the pre-service teachers for analyzing the proofs of the theorems used in the interviews. The pilot study was conducted with 20 pre-service teachers who volunteered to participate in the study; who were third-year students at the department of secondary mathematics teaching; and who took Analysis I, Analysis II and Analysis III courses. Since the analysis courses are based upon function and its properties, these pre-service teachers gained detailed information on function concept. During the pilot study stage, it was ensured that all pre-service teachers who were willing to participate in the research were included in the research. Accordingly, the aim was to perform more case studies for the theorems used in the interviews and to increase mistake-free problems.

The pilot study was conducted in a classroom environment in an hour on which the pre-service teachers did not have courses. The pre-service teachers were requested to choose two out of five theorems that were prepared on the concepts of one-to-one function and surjective function and to prove the theorems that they chose. The pilot study lasted for 50 minutes. Taking pre-service teachers' preferences into account, the theorems that they preferred the most were used as the interview problems of the actual study. Furthermore, the data obtained in the pilot study were analyzed separately by the writer and two mathematics educators, and the difficulties experienced by the pre-service teachers were identified. Then, each difficulty was discussed and a consensus was reached on the difficulties to be used in the actual study.

Research Group

The research was conducted with six pre-service teachers who volunteered to participate in the study and who were studying at the department of secondary mathematics teaching. The pre-service teachers were in the finishing stage of five-year secondary mathematics teaching undergraduate program. The purposive sampling method was used in selecting the participants. Patton (2002), defines the purposive sampling as the selection of the cases from which rich information can be obtained in terms of shedding light on the problems, the answers of which are sought in the study.

Data Collection

The data of the research were collected the semi-structured interviews. The interviews were prepared on the concepts of one-to-one function and surjective function. Problems which would allow the pre-service teachers to perform proof on the related concepts were determined by the researcher in each interview. Furthermore, the problems prepared on the designated concepts were revised by the researcher by taking the opinions of three mathematics educators who were field experts. This is because the content validity of measurement tools is subjective and based on expert opinion. The content validity of the activity problems was established in accordance with the conducted studies and expert opinions.

Before moving on to the interviews with the pre-service teachers, the researchers explained that the research was completely based upon the principle of voluntariness, and the pre-service teachers, who did not want to continue this study, could leave the study whenever they wished. Furthermore, the pre-service teachers were told that the interviews would be recorded with a camera. They were asked whether this condition constituted an inconvenience for them. Their permission was obtained on this matter. The researchers stated that the names of and information about the pre-service teachers would not be shared with anybody, and aliases would be used instead of their real names in the results of the research. The interviews, which were conducted with the pre-service teachers, lasted for approximately 25-30 minutes. All interviews were conducted in an environment where the first writer and the pre-service teachers could talk one-to-one. Furthermore, the information (definitions), which was going to be used by the pre-service teachers in performing proof, was given in the interview forms. The theorems that were asked in the interviews and the information that had to be used in proving these theorems are given below.

Theorem 1: If f and g functions are one-to-one, $g \circ f$ function is also one-to-one. Prove it.

In order for the pre-service teachers to prove Theorem 1, it is adequate to show that $g \circ f$ function is one-to-one using the one-to-one function definition. To do so, the pre-service teachers need to make use of the fact that f and g functions are one-to-one, and $g \circ f(x) = g(f(x))$ property of the compound function.

Theorem 2: If $g \circ f$ function is surjective, g function is also surjective. Prove it.

In order for the pre-service teachers to prove Theorem 2, they need to make use of the surjective function definition and the fact that $g \circ f(x) = g(f(x))$ function is surjective.

Analysis of the Data

The difficulties experienced by the pre-service teachers were identified by the researcher by benefiting from pilot study data and the literature in analyzing the interviews. Furthermore, the opinions of two mathematics educators were taken in order for the researcher to finalize the identified difficulties. In line with the opinions of these experts, the difficulties were finalized by the researcher. Each answer given by the pre-service teachers was analyzed according to the identified difficulties. For this reason, the answers of the pre-service teachers were analyzed in the scope of the "content analysis" model, which is considered to be a qualitative research method. This is because the content analysis makes it possible to give meaning to the obtained raw data; to form a certain framework; to arrange this data after the emerged case has become clear; and to reveal and concretize categories and codes (Patton, 2002). Moreover, the conducted content analysis was supported with qualitative and descriptive analysis by giving excerpts from the proofs performed by the pre-service teachers.

In pre-service teachers' proofs for the theorems related to one-to-one function and surjective function, it was found that they had the following difficulties: *determining how to begin the proof process (D1), the use of mathematical language and notations (D2), the use of definitions (D3), forming the setup of the proof (D4) and selecting elements from the set (D5)*. These difficulties and their indicators are presented in Table 1.

Difficulties	**Indicators**
Determining How to Begin the Proof Process (D1)	Difficulties experienced in determining how to begin the proof process
The Use of Mathematical Language and Notations (D2)	Difficulties experienced in using appropriate mathematical language and notations for the theorem statement
The Use of Definitions (D3)	Difficulties experienced in understanding the definitions that must be used in the proof process
Forming the Setup of the Proof (D4)	Difficulties experienced in determining how to proceed in the proof process
Selecting Elements from the Set (D5)	Difficulties experienced in selecting appropriate elements for the domain and codomain of the functions used in the proof process

Table 1: Difficulties experienced by the pre-service teachers in the proof process and the indicators of these difficulties

Findings

The findings obtained from the interviews conducted with the pre-service teachers are given in this section of the research. The findings on the concepts of one-to-one function and surjective function are presented in tables according to D1, D2, D3, D4 and D5 difficulties designated in the research.

Difficulties	**One-to-One Function**	**Surjective Function**
	Pre-Service Teachers	Pre-Service Teachers
D1	PT1, PT2, PT6	PT1, PT2, PT5
D2	PT1, PT2, PT3, PT4, PT6	PT1, PT2, PT3, PT5, PT6
D3	PT2, PT4, PT6	PT1, PT3, PT5, PT6
D4	PT2, PT6	PT1, PT5
D5	PT1, PT2, PT3, PT4, PT5, PT6	PT1, PT3, PT5, PT6

Table 2: Classification of the pre-service teachers' proofs processes regarding the concepts of one-to-one function and surjective function according to categories of difficulty

Concept of One-to-One Function

Pre-service teachers' theorem proofs regarding the concept of one-to-one function are descriptively presented below.

The Case of PT1

Interviewer: How would you proceed in proving this theorem?
PT1: [He reads the theorem statement]... Let both functions here be defined from ℝ to ℝ... Let me write them... Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$... In that case, let's write it like this in view of injectivity definition. Again, we can write that the images of different elements are different in f and g functions... [He is thinking]... Or I had better start directly from the compound function... That is because our operation could become a little complicated then. In that case, suppose that we take $g \circ f(x_1) = g \circ f(x_2)$ for $\forall\, x_1, x_2 \in \mathbb{R}$. If I manage to show that $x_1 = x_2$ from this statement, $g \circ f$ is one-to-one. Then, I can write these statements as $g\big(f(x_1)\big) = g\big(f(x_2)\big)$ from the compound definition. Now, I can make use of the fact that f and g functions are one-to-one... However, I wonder if I should make use of the fact that f is one-to-one or g is one-to-one first... [He is thinking]... Well... We can remove g first in this statement... Then, we can write $f(x_1) = f(x_2)$ since g is one-to-one. Similarly, we find $x_1 = x_2$ since f is one-to-one. In that case, $g \circ f$ is one-to-one according to my initial acceptance.

When PT1's proof regarding the concept of one-to-one function is examined, it is observed that he has D1, D2 and D5 difficulties. The pre-service teacher wanted to begin the proof process by firstly making use of the fact that f and g functions were one-to-one. However, when he further contemplated on the proof, he understood that he would not be able to reach the solution in this manner. Since the pre-service teacher decided that he could complete the proof process more comfortably by using the compound function, he formed the setup of the proof accordingly. However, PT1 considered f and g functions only on ℝ and selected the elements accordingly. On the other hand, it is observed that the pre-service teacher did not use any notation regarding the domain and codomain of $g \circ f$ function. Since he was not able to identify the domain and codomain of the compound function precisely, it cannot be understood clearly to which set that the elements he used belong.

The Case of PT2

PT2: [He reads the theorem statement]... If f and g are one-to-one, they fulfill the injectivity definitions. We can say $g(f(x))$ in $g \circ f(x)$ function... Now, let's see what we must do... Well... We must take an element again... Injectivity is as follows: I pair each element that I take from A with one element from B in $f\colon A \to B$ function... That is to say, no element must be left unpaired in A due to the condition of being a function, and I pair each element that I take from A with just one element from B. We can generalize this to $g \circ f$ set, too... Actually, I know what to do, but I don't know how to state it mathematically... Let me contemplate further... I can do this as follows... If I take two functions... Let $f\colon A \to B$ be a one-to-one function... If we use injectivity definition, we must say that $f(x_1) = f(x_2)$ when $x_1 = x_2$ and $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$ for $\forall x_1, x_2 \in A$. In view of this, $g(y_1) = g(y_2)$ when $y_1 = y_2$ and $g(y_1) \neq g(y_2)$ when $y_1 \neq y_2$ for $\forall y_1, y_2 \in B$. Now, I must take the compound of these... Hmm, how can I do that?... [He cannot decide what he must do]... [He is thinking]... Actually, it is a very simple proof, but I cannot remember it now. I can do this as follows: Let's say that $g \circ f(x_1) = g(f(x_1)) = g(y_1) = g(y_2) = g \circ f(x_2)$ when $x_1 = x_2$... Again, we can write $g \circ f(x_1) = g(f(x_1)) = g(y_1) \neq g(y_2) = g \circ f(x_2)$ when $x_1 \neq x_2$. That is because f and g functions are one-to-one. In view of this, I found $g \circ f(x_1) = g \circ f(x_2)$ when $x_1 = x_2$ and $g \circ f(x_1) \neq g \circ f(x_2)$ when $x_1 \neq x_2$. Therefore, $g \circ f$ is one-to-one.

When PT2's proof is examined, it is observed that he has D1, D2, D3, D4 and D5 difficulties. The pre-service teacher was not able to form the setup of the proof since he could not decide how to begin the proof process. In addition, he was not able to make use of one-to-one function definition that had to used in the proof. The pre-service teacher used unnecessary mathematical statements and notations since he was not able to understand one-to-one function definition. Furthermore, it was observed that the pre-service teacher was not able to understand on which set he was working since he was not able to form appropriate domain and codomain for $g \circ f$ function. Thus, it can be stated that the pre-service teacher was not able to form the proof and he got lost among unnecessary expressions.

The Case of PT3

PT3: [He reads the theorem statement]... This could be as follows... Let's use $x = y$ definition when $f(x) = f(y)$... Now, let $g \circ f(x) = g \circ f(y)$... $g(f(x)) = g(f(y))$ since $g \circ f(x) = g(f(x))$, $g \circ f(y) = g(f(y))$. Now, if f is one-to-one, [$x = y$ when $f(x) = f(y)$]. In view of this, $g(x) = g(y)$... Since g is also one-to-one, $x = y$. Thus, the proof is completed.

When PT3's proof regarding the concept of one-to-one function is examined, it is observed that he has D2 and D5 difficulties. The pre-service teacher correctly decided how to begin the proof process and he set up the proof accordingly. PT3 also correctly used one-to-one function definition in the proof process. It is observed that PT3 did not state the sets on which he worked and that he was not able to interpret $g \circ f$ function. It is observed that the pre-service teacher firstly made use of the fact that f function is one-to-one since he was not able to interpret $g(f(x)) = g(f(y))$ mathematical statement. Furthermore, it cannot be understood to which set that x, y elements used in the proof belong. Thus, it can be stated that PT3 focused on only the operations in the proof process and overlooked the conceptual infrastructure of the proof.

The Case of PT4

PT4: Let's translate the information given to us into mathematical language... $x = y$ when $f(x) = f(y)$... Again, it was stated that $x = y$ when $g(x) = g(y)$ [I used x and y here, too. It doesn't make a difference, though]. Now, we are looking for $x = y$ when $g \circ f(x) = g \circ f(y)$... Now, $g(f(x)) = g(f(y))$ since $g \circ f(x) = g(f(x))$ and $g \circ f(y) = g(f(y))$... If I take $f(x) = a$, $g(a) =$... [Here, $f(y) = a$... Well... I guess I have reached an impasse... He is thinking. He is checking what he has written]... Hmm... Now, if I take $f(y) = b$... Then, $a = b$ when $g(a) = g(b)$... Well... I was trying to show that $x = y$... [He is thinking]... In view of this, $f(x) = f(y)$ and $x = y$ if $a = b$. Thus, the proof is completed.

When PT4's proof is examined, it is observed that he has D2, D3 and D5 difficulties. The pre-service teacher correctly decided how to begin the proof process and he set up the proof accordingly. However, PT4 was not able to make use of one-to-one function definition as effectively as he wanted since he used unnecessary mathematical statements and notations in the proof process. Furthermore, his proof became meaningless due to excessive use of operations, as he was not able to understand to which set that the elements he used belonged. Thus, the proof of the pre-service teacher became substantially complicated and mathematically meaningless.

The Case of PT5

PT5: [He reads the theorem statement]... If it is stated that f and g are one-to-one, then, we say $x_1 = x_2$ when $f(x_1) = f(x_2)$, and similarly, $y_1 = y_2$ when $g(y_1) = g(y_2)$. Then, $g \circ f$ function... I will have to show that $x_1 = x_2$ taking $(g \circ f)(x_1) = (g \circ f)(x_2)$ into consideration... Now, we can write $g(f(x_1)) = g(f(x_2))$ by making use of compound operation property. Now, here, $f(x_1)$ gives us a y_1 value... That is because when we consider this according to domain, each element will have to have an image... Similarly, $f(x_2) = y_2$. In view of this, I reach $y_1 = y_2$ value when $g(y_1) = g(y_2)$. That is to say, I have seen that it is one-to-one.
Interviewer: Does starting with $(g \circ f)(x_1) = (g \circ f)(x_2)$ and reaching $y_1 = y_2$ value instead of $x_1 = x_2$ value affect the validity of the proof? If so, why?
PT5: [He is thinking]... I think it will not pose a problem when we consider it as a domain and codomain... In other words, it will not constitute a mistake in my opinion.

When PT5's proof is examined, it is observed that he has only D5 difficulty. The pre-service teacher correctly decided how to begin the proof process, mathematical statements and notations that he should use, the definition that he should use in the proof process and finally the setup of the proof. However, it cannot be understood to which set that the elements he used belong since he was not able to fully interpret $g \circ f$ function. Furthermore, since he incorrectly interpreted $g(f(x_1)) = g(f(x_2))$ mathematical statement, he was not able to fully make use of that statement. In addition, he was not able to clearly understand to which set that x, y elements used in the proof belong. Thus, it can be stated that PT5 overlooked the domain and codomain on which he worked.

The Case of PT6

PT6: First of all, since I will be using direct proof method, I have to check whether or not f and g fulfill the conditions. That is to say, $x = y$ for $f(x) = f(y)$ since f is one-to-one. Similarly, $x = y$ for $g(x) = g(y)$ since g is one-to-one, too. Here, we will show whether or not $x = y$ if $g \circ f(x) = g \circ f(y)$. I have written all of these as x, y. I guess it will not pose a problem... Now, let's take $g(f(x)) = g(f(y))$... [He is not sure of what he has written. He is again erasing what he has written]... I call the compound function h... That is to say, let $h = g \circ f$. In that case, in order to show that h is one-to-one, we will check whether or not $x = y$ if $h(x) = h(y)$... Yes... $g(f(x))$, f are one-to-one; g is also one-to-one... Here, I assign a value to $f(x)$... Then, let's take $f(x) = f(y) = a$ and $g(x) = g(y) = b$... In view of this, $g(f(x)) = g(f(y))$, and accordingly, $g(a) = g(a)$... This is not giving me the correct answer... What should I do?...
Interviewer: I wonder if you have begun the proof process incorrectly.
PT6: [He is thinking]... How can I begin it in any other way?
Interviewer: It can be the opposite of what you have done... That is to say, if we want to show that $g \circ f$ is one-to-one, we can begin by writing the statements that are necessary for $g \circ f$ to be one-to-one. Then, we can make use of the hypothesis.

PT6: [He is thinking]... Alright... Let's try one more time in this manner... [He is erasing what he has written]... Now, we will show that $g \circ f$ is one-to-one... $x = y$ if $g \circ f(x) = g \circ f(y)$. In that case, we should break it down... $g(f(x)) = g(f(y))$... Well, I wrote that a little while ago...
Interviewer: I think we can now make use of the fact that f and g are one-to-one.
PT6: Hmm... Yes, we can see that $f(x) = f(y)$ since g is one-to-one. Hmm, I have done just the opposite in the beginning... Similarly, $x = y$ since f is one-to-one. Hmm, I have now understood. That is the condition of being one-to-one, and we have found it. I have started it from the opposite way... I have understood it.

When PT6's proof is examined, it is observed that he has D1, D2, D3, D4 and D5 difficulties. The pre-service teacher was not able to form the setup of the proof and complete the proof since he could not decide how to begin the proof process. Furthermore, he used unnecessary and incorrect mathematical statements in the proof process. Therefore, he was not able to make use of the definition that he selected for the proof process. Apart from this, since he was not able to understand to which sets that the elements he used in the proof belonged, he mistook these elements for one another. PT6 was not able to complete the proof process, and he tried to complete it with the help of researcher's questions at the end.

Concept of Surjective Function

Pre-service teachers' theorem proofs regarding the concept of surjective function are descriptively presented below.

The Case of PT1

Interviewer: How would you proceed in proving this theorem?
PT1: If $g \circ f$ is surjective, [He is thinking]... Since we are trying to show that only g is surjective, we must find that $a \in \mathbb{R}$ when $g(a) = b$ for $\forall b \in \mathbb{R}$. Now, after writing that... [He is thinking]... Let's see what we can write... No information was given on the fact that f is surjective... Let me think about what we can do... [He is thinking]... I wonder what would happen if we drew its graph... $g \circ f$ is surjective... If this is surjective, the surjectivity of g [He is thinking]... No, I cannot reach the result in this manner... I mean if f function had also been surjective, I might have continued the operation; but I cannot go on with the way it is.

When PT1's proof regarding the concept of surjective function is examined, it is observed that he has D1, D2, D3, D4 and D5 difficulties. The pre-service teacher was not able to decide what to do in the proof process. He argued that f function had to be surjective in order to continue the proof process. Not only this, but he was also not able to determine the mathematical statements and notations that he had to use in the proof. He was not able to make use of surjective function definition since he could not identify the domain and codomain of $g \circ f$ function. Therefore, PT1 stated that he could not perform the proof for the theorem. Since domain and codomain are very important in surjective function definition, it can be stated that the pre-service teacher's failure to identify the domain and codomain of $g \circ f$ function played a key role in his failure to continue the proof process.

The Case of PT2

PT2: Since $g \circ f$ is surjective and an element will definitely have an image in f function, g must necessarily be surjective... [How can I do this?]... There is $a \in A$ and when $g \circ f(a) = c$ for $\forall c \in C$... This shows that... $g(a) = c$... [He is thinking]... Actually, we mustn't say that $a \in A$... Let it be as follows... There is a when $g \circ f(a) = c$. Here, let $f(a) = b \in B$... I mean I don't know whether f is surjective, so I mustn't deal with f... After all, I see that it is not required logically. [He is thinking]... I mean I don't know how to express this... Indeed, the easiest way would be to say that the theorem is insufficient, but that would be absurd...☺ Hmm... If $f(a) = b \in B$, then $g(b) = c$... We already know that $g: B \to C$ is a function... Thus, we have obtained a c for each a. I mean I can intuitively see that very easily, but I have had considerable difficulty in expressing it.

When PT2's proof is examined, it is observed that he has D1 and D2 difficulties. The pre-service teacher experienced difficulty in deciding how to begin the proof process. However, when he decided how to begin the proof process, he formed the setup of the proof. Apart from this, PT2 was able to identify to which sets that the elements he used in the proof had to belong since he could identify domain and codomain of, $g \circ f$ function. However, as he stated himself, the pre-service teacher experienced difficulty in completing the proof since he used unnecessary mathematical statements and notations.

The Case of PT3

PT3: $g \circ f$ was given as surjective... Therefore, there is a when $g \circ f(a) = b$. Now, in view of this, $g(f(a)) = b$. Here, we don't know whether an element exists for $f(a)$ because it is not surjective... However, since f is a function, $f(a)$ is necessarily the image of an element like c. Therefore, there is a when $f(a) = c$. In view of this, we find that there is c element when $g(c) = b$. Then, g is surjective.

When PT3's proof is examined, it is observed that he has D2, D3 and D5 difficulties. The pre-service teacher was able to decide how to begin the proof process and the setup of the proof. However, the domain and codomain of the functions cannot be understood since the mathematical statements and notations that he used in the proof are not clear. For that reason, PT3 experienced difficulty in identifying to which sets that the elements he used in the proof belonged. Thus, he was not able to make use of surjective function definition as effectively as he wanted. It can be also stated that PT3 focused on the operational section of the proof and overlooked the conceptual infrastructure.

The Case of PT4

PT4: Let me write the functions first... Let $f: A \to B$, $g: B \to C$ and $g \circ f: A \to C$... Here, we must find that $b \in B$ when $g(b) = c$ for $\forall c \in C$. Due to the theorem, we know that $g \circ f$ is surjective. That is to say, there is $a \in A$ when $g \circ f(a) = c$ for $\forall c \in C$. Now, let's break down $g \circ f$ compound function... We know that $g \circ f(a) = g[f(a)]$... We are trying to find $b \in B$ when $g(b) = c$ for $\forall c \in C$. Now, if I use the same notations, I find that $b \in B$ when $g(b) = c$ since $g \circ f(a) = g[f(a)] = c$ and $f(a) = b$ (every element in A goes to B since $f: A \to B$ is a function). Thus, the proof is completed.

PT4 fully performed the proof regarding the concept of surjective function. While performing the proof, PT4 did not experience any of the difficulties identified in the research. He correctly began the proof process and appropriately selected the mathematical statements and elements he used in the proof for the theorem statement.

The Case of PT5

PT5: $g \circ f$ function is surjective... [He is thinking]... Here, we know that $g \circ f(x) = g(f(x))$. Here, it had to be given that f is surjective. That is because I cannot move on to the subjectivity of g function if f is not surjective. In view of this, if f is surjective, $g \circ f(x) = g\big(f(x)\big) = g(y)$. Then, we see that g is surjective.

When PT5's proof is examined, it is observed that he has D1, D2, D3, D4 and D5 difficulties. The pre-service teacher argued that f function had to be given as surjective since he was not able to decide what to do in the proof process and continue the proof process. PT5 used incorrect mathematical inferences and statements since he continued his proof process by accepting that f function was surjective. The pre-service teacher wrote meaningless mathematical statements instead of performing the theorem proof.

The Case of PT6

PT6: If $g \circ f$ is surjective, we know that... If $g \circ f(a)$ is surjective, there is $a \in A$ when $g \circ f(a) = b$... We know that... Furthermore, we will try to show that g is surjective... [He is thinking]... Now, I need an element like $f(a) = c$ to reach the surjectivity of g from $g(f(a))$ statement... That is to say, I should be able to do this without considering that f is surjective... In other words, I will only need to make use of the fact that g is surjective... Therefore, we must find that $f(a) = c$ when $g(f(a))$... Since f is a function, there is such an element. Thus, g is surjective.

When PT6's proof is examined, it is observed that he has D2, D3 and D5 difficulties. The pre-service teacher was able to decide how to begin the proof process and what to do later. However, he was not able to fully state the domain and codomain that he had to use in the proof. Therefore, the mathematical statements and notations that he used fell short. Furthermore, he was not able to fully explain how he reached the result of the proof. Thus, it can be stated that he experienced difficulty in understanding surjective function definition.

Discussion

The results of the research show that the difficulties experienced by the pre-service teachers while performing proofs regarding the concepts of one-to-one function and surjective function are grouped under five categories (D1, D2, D3, D4 and D5). It was found that the difficulties experienced by the pre-service teachers in proofs were as follows: determining how to begin the proof process (D1), the use of mathematical language and notations (D2), the use of definitions (D3), forming the setup of the proof (D4) and selecting elements from the set (D5). On the other hand, it was observed that the difficulties experienced by the pre-service teachers in proofs rather focused on D2 and D5 categories.

The basic reason for the difficulties experienced by the pre-service teachers in D1 category is that they failed to understand the relationship between what was given (hypothesis) and what was asked in the theorem statement. Thus, the pre-service teachers who experienced difficulty in D1 category were not able to decide how to begin the proof of theorem statement. In fact, the pre-service teachers were aware that the provision section of the theorem was supposed to be proven and that they had to begin the proof process accordingly. However, they were not able to decide how to make use of the hypothesis in order to find what was asked in the theorem. For instance, the following statements support this condition: *Injectivity is as follows: I pair each element that I take from* A *with one element from* B *in* $f\colon A \to B$ *function... That is to say, no element must be left unpaired in* A *due to the condition of being a function, and I pair each element that I take from* A *with just one element from* B. *We can generalize this to* $g \circ f$ *set, too... Actually, I know what to do, but I don't know how to state it mathematically...*

The mathematical language and notations used by the pre-service teachers are important in order for them to transfer the proof that they perform to their students in a clear manner and in order for the proofs to be understood (Güler & Dikici, 2013). Therefore, the difficulties experienced by the pre-service in using the appropriate mathematical language and notations for the mathematical statements within the theorem statement were studied in D2 category. It was observed that D2 is one of the most frequently experienced difficulties by the pre-service teachers while performing both proofs. It was found that all pre-service teachers, excluding PT5, experienced difficulty in this category in the proof regarding the concept of one-to-one function whereas all pre-service teachers, excluding PT4, experienced difficulty in this category in the proof regarding the concept of surjective function. It can be stated that the difficulties in this category generally resulted from the use of insufficient or unnecessary statements. Furthermore, it was generally observed that the pre-service teachers did not name the sets on which they worked and they did not use appropriate notations for these sets. It was observed that difficulties were experienced in the later parts of their proofs resulting from the complexity of notations. This result obtained in the research supports the opinion that the pre-service teachers experience difficulty in expressing the concepts that they use in proofs (Güler & Dikici, 2013).

The difficulties experienced in D3 category in proofs result from the fact that the pre-service teachers were not able to fully understand what the definitions that they used in their proofs meant. The fact that the concepts within the research have more than one definition requires the pre-service teachers to identify any of the definitions of these concepts and perform the proof accordingly. It can be stated that the difficulties experienced by the pre-service teachers in the use of definitions in proofs resulted from their failure to decide which definition to use for the concept, failure to name the sets on which they work, and failure to select elements for these sets by using appropriate notations. It was observed that the pre-service teachers were not able to make use of the definitions due to the use of insufficient or unnecessary notations in the proof especially regarding the concept of surjective function. As required by surjective function definition, appropriate notations must be used for the domain, codomain and image set. It was observed that the pre-service teachers who did not pay attention to this condition were not able to fully make use of the definitions.

The difficulties experienced in D4 category in proofs result from the fact that the pre-service teachers were not able to precisely determine the steps that they must follow. As a matter of fact, it is important for the pre-service teachers to clarify the logic of the proof and the steps that they must follow in the proofs that they will perform (Güler & Dikici, 2013). Therefore, the pre-service teachers need to set up the proofs according to the theorem statement. That is because proofs are built upon a certain system, and we make use of the previous step in each step. The pre-service teachers who participated in the research generally know how to proceed in the proofs. However, it was observed that the pre-service teachers having D1 difficulty generally experienced difficulties in setting up their proofs. Therefore, they were not able to effectively use the information in the theorem statement.

When the answers given by the pre-service teachers were examined according to D5 difficulty, it was observed that majority of the pre-service teachers experienced difficulty in selecting appropriate elements for the domain and codomain of the functions that they used in the proofs. As a matter of fact, it was observed that all pre-service teachers had D5 difficulty in the proof regarding the concept of one-to-one function whereas the pre-service teachers, excluding PT2 and PT4, had D5 difficulty in the proof regarding the concept of surjective function. It can be stated that the reason for the pre-service teachers' failure to select appropriate elements for the domain and codomain of the functions on which they worked is the fact that they did not express these sets mathematically. Thus, it was observed that the pre-service teachers, who experienced difficulty in D2 category, generally had D5 difficulty. The pre-service teachers were not sure of the correctness of the results that they reached since they did not express from which set they took the elements that they selected in their proofs. This is because the concepts of one-to-one function and surjective function were defined on domain and codomain. It was found that the pre-service teachers were not careful in this respect and consequently experienced difficulty in their proofs.

In the research, it was observed that the difficulties experienced by the pre-service teachers in the proofs regarding the concepts of one-to-one function and surjective function were interrelated. It was found that the pre-service teachers who had D1 difficulty generally experienced D4 difficulty whereas the pre-service teachers who had D2 difficulty generally experienced D3 and D5 difficulties. In this respect, the results of the research show that to decide how to begin the proof process is important in order to complete the proof and plan the proof. On the other hand, it can be stated that use of correct mathematical language and notations constitutes the conceptual infrastructure of the proof. Therefore, it can be stated that the use of mathematical language and notations affects the selection of appropriate elements for domain and codomain in especially the proofs regarding the concept of function. This result supports the opinion that the majority of the students do not know how to prove; how to begin the proof process; the conceptual knowledge and definitions that they must use during the proof process; and how to use them (Weber, 2001). Furthermore, regarding the difficulties experienced by the pre-service teachers while performing proof, it was observed that the results obtained in the research are parallel with the results of other research in the literature that sets forth the existence difficulties in determining how to begin the proof process (Güler, 2013; Güler & Dikici, 2013; Moore, 1990; 1994; Selden & Selden, 2003), expressing the definitions used in the proof (Knapp, 2005; Bayazıt, 2009) and the use of mathematical language and notations (Moore, 1994; Selden & Selden, 2007). Moreover, in view of the research result, the difficulty of selecting elements from the set was included in the literature on the difficulties experienced by the pre-service teachers while performing proof.

In this study, the difficulties experienced by the pre-service teachers were examined according to the concepts of one-to-one function and surjective function. Therefore, this study is limited to the proofs of the theorems regarding these two concepts. Thus, different difficulties can be observed in the theorem proofs related to different concepts. In this context, different concepts can be researched in the studies that will be conducted on the function concept in the future. Consequently, potential difficulties can be compared, and the pre-service teachers can be informed on this issue. Furthermore, by discussing the difficulties that they experience in the proofs with them, the pre-service teachers can be made to realize their own difficulties and those experienced by their friends. Accordingly, a basis can be formed to design classroom activities for eliminating the difficulties experienced by pre-service teachers.

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